G-Vector Bundles over G-Manifolds (Geometry of Manifolds)

MATSUNAGA, HIROMICHI

数理解析研究所講究録
158巻 1972年 93-101

G-VECTOR BUNDLES OVER G-MANIFOLDS

OSAKA CITY UNIVERSITY HIROMICHI MATSUNAGA

At first I want to talk about some of my works, which I have stated in other seminar, but it is my starting point, so let me do that.

Definition. Let G be a compact Lie group. A real or complex vector bundle $E \longrightarrow X$ is a G-vector bundle if and only if

(1) E and X are G-spaces,

(2) the projection p is an equivariant map,

(3) for each $x \in X$ and $g \in G$, the action $g: E_x \longrightarrow E_{gx}$ is linear.

Let M be a compact G-manifold with orbit type $((H),(K))$, where H, K are closed subgroups of G with $H \subset K$.

Definition (K.Jänich). A G-manifold M is special if and only if, denoting by V_x the normal space to the orbit through x for each $x \in M$, the slice representation $G_x \longrightarrow \mathrm{Aut}\,(V_x)$ admits a decomposition $V_x = F_x \oplus W_x$ such that $G_x \,|F_x$ = the identity of F_x and $G_x \,|S(W_x)$ (the unit sphere in W_x) is transitive.

We use some notations. $M_{(K)}$: the union of all singular orbits, which is a closed submanifold of M. $M_1 = \overline{M-N(M_{(K)})}$: the closure of an invariant tublar neighborhood of $M_{(K)}$. $\widehat{\mathrm{Vect}}_K$: the family of all K-vector bundles.

The projection $p : \partial N(M_{(K)}) \longrightarrow M_{(K)}$ induces a diffeomorphism $p' : \partial\pi(M_1) \longrightarrow \pi(M_{(K)})$, where π is the projection $M \longrightarrow M/G$. For a pair $(F,E) \in \widehat{Vect}_K(\pi(M_{(K)}) \times \widehat{Vect}_H(\pi(M_1))$, let $\alpha_H : p'^*r^*F \longrightarrow E\,|\partial\pi(M_1)$ be an isomorphism of H-vector bundles, where we denote by r^* the restriction $\widehat{Vect}_K \longrightarrow \widehat{Vect}_H$.

Definition 1. Two triples (F,E,α_H) and $(\bar{F},\bar{E},\bar{\alpha}_H)$ is equivalent if and only if there exists a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "p'^*r^*"] \arrow[d, "\rho_K"] & p'^*r^*F \arrow[r, "\alpha_H"] \arrow[d, "\rho_{H,K}"] & \partial E \arrow[r, phantom, "\subset"] \arrow[d, "\partial\varphi_H"] & E \arrow[d, "\varphi_H"] \\
\bar{F} \arrow[r, "p'^*r^*"] & p'^*r^*\bar{F} \arrow[r, "\bar{\alpha}_H"] & \partial\bar{E} \arrow[r, phantom, "\subset"] & \bar{E},
\end{tikzcd}$$

where $\rho_K(\varphi_H)$ is an isomorphism of K(H)-vector bundles, and $\rho_{H,K}$, $\partial\varphi_H$ are restrictions of them.

Theorem. Under the condition

(C) $N(H) = H \times \Gamma(H)$, $\Gamma(H) = N(H)/H$, $N(K) = K \times \Gamma(K)$, $\Gamma(K) = N(K)/K$, $\Gamma(K) \subset \Gamma(H) \subset G$, we have an isomorphism of semi-groups

$$Vect_G(M) \approx \{(E,F,\alpha_H)\}/(\sim),$$

where we mean by $/(\sim)$ the classification due to Definition 1.

By the theorem and an analogy of the Atiyah-Bott's proof of Bott periodicity, and using H.Minami's result about $R(O(n))$, I have obtained $K_{O(n)}(W^{2n-1}(d)) \cong R(O(n-1))$. These results are to be appeared in Osaka Journal of Mathematics. ([1])

In the determination of the K_G-group, I have essentially used the splitting of the normalizer $N(I_r \times O(n-r)) = O(r) \times O(n-r)$, $r = 1,2$. The condition ($\mathcal{C}$) is too restrictive for applications, so I want to improve it.

By a technical reason I take right actions. Let M be a compact right G-manifold with just one orbit type (H). Denote by Γ the factor group $H \backslash N(H)$, then we have a differentiable principal bundle

$$(1) \quad \Gamma \longrightarrow M_H \longrightarrow M_H/\Gamma \quad ,$$

where $M_H = \{ x \in M \; ; \; G_x = H \}$ and the G-manifold M is the total space of the associated fiber bundle, that is $M \cong M_H \times_\Gamma (H\backslash G)$ as a G-manifold. By G.Segal

$$\mathrm{Vect}_G(M) \approx \mathrm{Vect}_{N(H)}(M_H).$$

(Under the condition ($\mathcal{C}$), $\mathrm{Vect}_{N(H)}(M_H) \approx \mathrm{Vect}_H(M_H/\Gamma)$.) Since M_H is a compact differentiable manifold, there exists an open covering $\{U_i\}$ of M_H/Γ such that U_i is deformable to a point x_i of U_i and a Γ-equivalence $\varphi_i : U_i \times \Gamma \longrightarrow M_H | U_i$ for each i. For any N(H)-vector bundle $E \longrightarrow M_H$, there are N-vector bundles $E_i \longrightarrow U_i$ with $\varphi_i{}^*E_i \cong E \mid \varphi_i(U_i \times \Gamma)$, $(N = N(H))$. Further we have an isomorphism of N-vector bundles $E_i \overset{N}{\cong} U_i \times (E|x_i \times \Gamma)$, because $U_i \times \Gamma$ is N-deformable to $x_i \times \Gamma$, where N-action over Γ is given by the projection $q : N \longrightarrow \Gamma = H\backslash N(H)$. On the other hand, by G.Segal $E_i|x_i \times \Gamma \overset{N}{\cong} V_i \times_H N$, where $V_i = E_i | x_i \times (e)$,

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which is an H-module. Thus we have an isomorphism of N-vector bundles,

$$\begin{array}{ccc} U_i \times V_i \times_H N & \xrightarrow[\approx]{\Psi_i} & E \mid \varphi_i(U_i \times \Gamma) \\ \Big\downarrow p_i & & \Big\downarrow p \\ U_i \times \Gamma & \xrightarrow[\approx]{\varphi_i} & \varphi_i(U_i \times \Gamma) \end{array}$$

Denote the N-equivalence $\Psi_j^{-1}\Psi_i : (U_i \cap U_j) \times V_i \times_H N \longrightarrow (U_i \cap U_j) \times V_j \times_H N$ by $\Psi_j^{-1}\Psi_i(x,[v,n]) = (x, G_{ji}(x)[v,n])$, then with respect to the CO-topology of $\mathrm{Iso}_N(V_i \times_H N,\ V_j \times_H N)$, G_{ji} is a continuous map for each i,j. By the usual verification (Part 1.[3]), we get the next propositions.

Proposition 1. For any N-vector bundle $E \longrightarrow M_H$,

$$E \overset{N}{\approx} \left[\textstyle\bigcup_i U_i \times V_i \times_H N\right] \Big/ (G_{ji}).$$

Proposition 2. **Two N-vector** bundles (E, G_{ji}) and (E', G'_{lk}) are equivalent if and only if there exist continuous maps $\bar{G}_{ki} : U'_k \cap U_i \longrightarrow \mathrm{Iso}_N(V_i \times_H N\ ,\ V'_k \times_H N)$ with the property

$$\bar{G}_{kj} G_{ji} = \bar{G}_{ki} \quad \text{on } U'_k \cap U_j \cap U_i\ , \quad G'_{lk}\bar{G}_{kj} = \bar{G}_{lj} \quad \text{on } U'_l \cap U'_k \cap U_j$$

Now to proceed much more, we consider the case which satisfies the condition

$$(\delta)\quad N(H) = H.\Gamma \ : \text{semi-direct product.}$$

For example, take SO(n) as G and $I_r \times SO(n-r)$ as H, then $N(I_r \times SO(n-r)) = SO(n-r).O(r)$. In fact the section $O(r) \longrightarrow N(I_r \times SO(n-r))$ is given by

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$$O(r) \ni A \longrightarrow \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & \det A & & \\ 0 & 0 & 1 & \\ \vdots & & & \ddots \\ 0 & 0 & & 1 \end{pmatrix} \in N.$$

The same property can be verified for $G = SU(n)$ and $H = I_r \times SU(n-r)$, these cases happen actually (Chap. 4, [2]).

Define an H-action over $V_i \times \Gamma$ by $(v, \gamma)h = (v.h^{\gamma}, \gamma)$, where $h^{\gamma} = \gamma h \gamma^{-1}$, then under the condition (δ), we have

$$V_i \times \Gamma \overset{N}{\approx} V_i \times_H N.$$

Then we can represent G_{ji} as follows :

$$\Psi_j^{-1}\Psi_i(x, v, \gamma) = (x,\ g_{ji}(x)(v),\ \gamma_{ji}(x)\gamma),$$

where γ_{ji} is the transition function of the principal bundle (1). We can prove that $g_{ji} : U_i \cap U_j \longrightarrow Iso(V_i, V_j)$ is continuous and

$$(*) \quad g_{ji}(x)(vh) = [g_{ji}(x)(v)]h^{\gamma_{ji}(x)}.$$

Thus in the semi-direct product case, Proposition 2 can be stated as

Proposition 2.δ. (E, g_{ji}, γ_{ji}) is N-equivalent to $(E', g'_{lk}, \gamma'_{lk})$ if and only if there exist continuous maps $\bar{g}_{ki} : U_i \cap U'_k \longrightarrow Iso(V_i, V'_k)$ with the property

$$\bar{g}_{ki}(x)(vh) = [\bar{g}_{ki}(x)(v)]h^{\bar{\gamma}_{ki}(x)},$$

$$\bar{g}_{kj}(x)g_{ji}(x) = \bar{g}_{ki}(x), \quad g'_{lk}(x)\bar{g}_{kj}(x) = \bar{g}_{lj}(x),$$

where $\bar{\gamma}_{ki}$ is the equivalence between γ_{ji} and γ'_{lk}.

By Proposition 2.$\mathcal{S}$, we define an equivalence relation of coordinate vector bundles $[\cup U_i \times V_i] / (g_{ji}, \gamma_{ji})$ over M_H/Γ with the property (*). We will call these bundles local H-vector bundles. Denote by $\mathrm{Vect}_{H\gamma}(M_H/\Gamma)$ the semi-group of equivalence classes. Then we have

Theorem. Let M be a G-manifold with just one orbit type (H), then under the condition ($\mathcal{S}$),

$$\mathrm{Vect}_G(M) \approx \mathrm{Vect}_{N(H)}(M_H) \approx \mathrm{Vect}_{H\gamma}(M_H/\Gamma).$$

Examples.

Consider the standard m-sphere $S^m = D_1^m \cup D_2^m$ (the union of the upper and lower hemi-spheres. Let $\gamma_{ji} : D_i^m \cap D_j^m \longrightarrow \Gamma$ be transition functions of a principal bundle $\Gamma \longrightarrow P \longrightarrow S^m$. For any local H-vector bundle E, $D_i^m \times V_i \approx E|D_i^m$, where V_i is an H-module for i = 1,2. We fix a point x_0 in $S^{m-1} = D_1^m \cap D_2^m$. By an appropriate choice of $(\bar{\gamma}_{ki}, \bar{g}_{ki})$, we can get a local H-vector bundle $(\gamma'_{ji}, g'_{ji}, D_i^m \times V_i)$ which is equivalent to $E = (\gamma_{ji}, g_{ji}, D_i^m \times V_i)$, and $g'_{12}(x_0)$ = the identity of $V_1 = V_2$ as a vector space and $\gamma'_{12}(x_0)$ = the unite of Γ. This is a usual normal form of a vector bundle over S^m (Part II, [3]). For each $v \in V_1$,

$$v \cdot {}_1h = g'_{12}(x_0)(v \cdot {}_1h) = [g'_{12}(x_0)(v)] \cdot {}_2h^{\gamma'_{12}(x_0)} = v \cdot {}_2h,$$

where we denote by ${}_ih$ the action in V_i. Thus we have $V_1 = V_2$ as an H-module. Now we investigate the case $G = SO(n)$, $H = I_r \times SO(n-r)$.

(I) Case $m \geq 2$.

Since S^{m-1} is connected, then $g'_{12}(S^{m-1}) \subset SO(r) \subset O(r)$ and $SO(n-r).SO(r) = SO(n-r) \times SO(r)$ direct product, further $SO(r) \times I_{n-r}$-action on $I_r \times SO(n-r)$ by the conjugacy is trivial. Thus we have

$$\mathrm{Vect}_H \gamma(S^m) \cong \mathrm{Vect}_H(S^m).$$

(II) Case $m = 1$.

We can prove the next lemma by the normal form technique.

Lemma $\mathrm{Vect}_H^{\mathbb{C}} \gamma(S^1) \approx \hat{H}^{\gamma}$ the semi-group of isomorphism classes of γ-invariant complex H-modules.

Suppose to be $\gamma_{12}(\pm 1) = \begin{pmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$. Set $n-r = 2s$ or $2s+1$.

We need a well known formula for the complex representation rings. Let T be the standard maximal torus of $SO(n-r)$, then $R(T) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_s, \alpha_s^{-1}]$, where α_k is the representation

$$\mathrm{Diag}\left\{\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \cdots \begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix}\right\} \longrightarrow e^{2\pi i \theta_k},$$

for $k = 1, 2, \ldots, s$. Then we have $R(SO(2s)) = \mathbb{Z}[\lambda^1, \ldots, \lambda^s_+, \lambda^s_-]$ with a relation $(\lambda^s_+ + \lambda^{s-2} + \ldots)(\lambda^s_- + \lambda^{s-2} + \ldots) = (\lambda^{s-1} + \lambda^{s-3} + \ldots)^2$, where $\lambda^k = \sigma^k[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_s, \alpha_s^{-1}]$ k-th elementary symmetric function and $\lambda^s_{\pm} = \sum_{\substack{1 \leq i_1 \leq \ldots \leq i_s \leq s \\ \varepsilon_1 \cdots \varepsilon_s = \pm 1}} \alpha_{i_1}^{\varepsilon_1} \cdots \alpha_{i_s}^{\varepsilon_s}$.

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By the relation

$$\left(\begin{array}{cccc|cccc} -1 & & & & & & & \\ & 1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & & & & & \\ \hline & & & & -1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right)\left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ \hline & 0 & & (a_{ij}) \end{array}\right)\left(\begin{array}{cccc|cccc} -1 & & & & & & & \\ & 1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & -1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \end{array}\right)$$

$$= \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & \ddots & & & & \\ \hline & & & a_{11} & -a_{12} & \cdots & -a_{1n} \\ & & & -a_{21} & & & \\ & 0 & & \vdots & & & \\ & & & -a_{n1} & & & \end{array}\right)$$

we have

$$(\alpha_1)^{\gamma_{12}(-1)} = \alpha_1^{-1} \quad ; \quad (\alpha_k)^{\gamma_{12}(-1)} = \alpha_k \, , \; k = 2, 3, \ldots, s$$

$$; \quad (\lambda^k)^{\gamma_{12}(-1)} = \lambda^k \, , \; k = 1, 2, \ldots, s \quad ; \quad (\lambda^s_{\pm})^{\gamma_{12}(-1)} \neq \lambda^s_{\pm} \, ,$$

and so

$$K(\mathrm{Vect}^C_H \gamma(S^1) = Z\,[\lambda^1, \ldots, \lambda^s] \subsetneq R(SO(n-r)).$$

When we treat the manifold $W^{2n-1}(d)$ as an $SO(n)$-manifold,

$W^{2n-1}(d)_{SO(n-1)} = S^1 = \{ (z_0, z_1) \; ; \; z_0^d + z_1^2 = 0, \; |z_0| = |z_1| = 1 \}$.

If d is odd, then the double covering $S^1 \longrightarrow S^1/O(1)$ is non trivial, and so $\gamma_{12} : S^0 \longrightarrow O(1)$ is surjective. Hence if n, d odd together, then

$$K_{SO(n)}(W^{2n-1}(d)_{(SO(n-1))}) \cong Z\,[\lambda^1, \ldots, \lambda^s] \subsetneq R(SO(n-1)).$$

References

[1] H.Matsunaga : $\tilde{K}_G$-groups and invariant vector fields on special G-manifolds, to appear in Osaka Journal of Mathematics.

[2] Wu-chung Hsiang and Wu-yi Hsiang : Differentiable actions of compact connected classical groups I , Amer. J. of Math. 139 (3), 1967.

[3] N.E.Steenrod : The topology of fiber bundles, 1951, Princeton.